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NUMERICAL SOLUTION OF STOCHASTIC EPIDEMIOLOGICAL MODEL: CASE OF SIR MODEL

EL BERRAI IMANE

Laboratory of Analysis, Modeling and Simulation, Department of Mathematics and Computer Sciences,
Faculty of Sciences, Ben M'Sik-Casablanca, Hassan II University Mohammedia-Casablanca Morocco

ABSTRACT

In this paper we extend the SIR epidemic model, considered in [1], from a deterministic framework to a stochastic one, and we formulate it as a stochastic differential equation. We present a Picard iteration method of proving verification theorem for the existence and uniqueness of the solution in global behavior. As a side effect of this study, we move one the development of the numerical treatment to solve the considered problem using the Milstein's scheme. A comparative numerical study is done using some values of the intensity of the stochastic environment around the endemic equilibrium.

KEYWORDS: SIR Model, Numerical Simulation, Stochastic Process, Picard Iteration, Brownian Motion, Stochastic Differential Equation

1. INTRODUCTION

In 1927, 1932 and 1933 W.O. Kermack and A.G. McKendrick published a series of papers titled Contributions to the mathematical theory of epidemics. Those papers are often seen as the basis of further research in mathematical (especially deterministic) modelling of the spread of infectious diseases. The first three papers of Kermack and McKendrick are reprinted in [10]. In the book of Diekmann and Heesterbeek [11] some of the results of Kermack and McKendrick and other deterministic models are presented and explained.

Other deterministic epidemiology models were then developed in papers by Ross, Ross and Hudson, Martini, and Lotka [5][8][9]. Mathematical epidemiology seems to have grown exponentially starting in the middle of the 20th century (the first edition in 1957 of Bailey's book [7] is an important landmark), so that a tremendous variety of models have now been formulated, mathematically analyzed, and applied to infectious diseases.

Allen [4] discusses a stochastic model of the above SIS epidemic model. However this is done by constructing a stochastic differential equation (SDE) approximation to the continuous time. The latter is obtained by assuming that events occurring at a constant rate in the deterministic model occur according to a Poisson process with the same rate. McCormack and Allen [12] construct a similar SDE approximation to an SIS multihost epidemic model and explore the stochastic and deterministic models numerically. Other previous work on parameter perturbation in epidemic models seems to have concentrated on the SIR model. Tornatore, Buccellato and Vetro [6] discuss an SDE SIR system with and without delay with a similar parameter perturbation. However this is not the only way to introduce stochasticity into the model.

The population is divided into three distinct classes: the susceptible S, "healthy individuals who can catch the disease", the infected I , "those who have the disease and can transmit it", and the removed R, "individuals who have had the disease and are now immune to the infection" (or removed from further propagation of the dis- ease by some other means). Schematically, the individual goes through consecutive states S → I → R. Such models are often called the SIR models.

2. STOCHASTIC STUDY

The SIR Model is used in the modeling of infectious diseases by computing the amount of people in a closed population that are susceptible, infected, or recovered at a given period of time.

The model is also used by researchers and health officials to explain the increase and decrease in people needing medical care for a certain disease during an epidemic. The SIR model is the basis for other similar models. The SI model, also known as the SIS model, is the model where once a person is no longer infectious, this per- son becomes susceptible once again. The common cold can be modeled with the SI model. There is also the SEIR model, where people are categorized as susceptible, exposed, infected, or recovered. The SIR model can be adjusted to include variation due to seasonal changes as seen by Bauch and Earn.[13]

2.1. Stochastic Differential Equation for SIR Model

we consider the SIR model given by :

$$\begin{cases} \dot{S} = r_c S\left(1 - \frac{S}{k}\right) - \frac{\alpha S I}{1 + aI} \\ \dot{I} = \frac{\alpha S I}{1 + aI} - \gamma I \\ \dot{R} = \gamma I - \delta R \end{cases} \tag{2.1}$$

The model has a susceptible group designated by S, an infected group I, and a recovered group R with permanent immunity, rc is the intrinsic growth rate of susceptible, k is the carrying capacity of the susceptible in the absence of infective, α is the maximum values of per capita reduction rate of S du to I, a is half saturation constants, γ is the natural recover rate from infection and δ is the death rate of recovered populations.

In the filtered probability space (Ω, F , (Ft)t≥0 , P), we consider the Stochastic version of deterministic SIR model (2.1), given by :

$$\begin{cases} dS_t = \left(r_c S\left(1 - \frac{S}{k}\right) - \frac{\alpha S I}{1 + aI}\right)dt + \sigma_S F(S, I, R)dW_t^S \\ dI_t = \left(\frac{\alpha S I}{1 + aI} - \gamma I\right)dt + \sigma_I G(S, I, R)dW_t^I \\ dR_t = \left(\gamma I - \delta R\right)dt + \sigma_R H(S, I, R)dW_t^R \end{cases} \tag{2.2}$$

with the initial conditions S(0) = S0 , I (0) = I0 , R(0) = R0 , and S, I and R represent the number of susceptible, infective and recovered individuals, respectively. σS F (S, I , R), σI G(S, I , R) and σR H (S, I , R) are the diffusion coefficients with σi (i = S, R, I) are real constants and known as the intensity of environmental fluctuations, Wi (i = S, R, I) are independent standard Brownian motions.

The aim of the following subsection is to study the existence and uniqueness of considered problem (2.2), than to present the numerical resolution. Further- more we complete our study by giving numerical resolution for different values of σk with {k = S, I , R}.

2.2. Existence and Uniqueness of Solution

2.2.1. Global Behavior

Throughout the rest of this chapter, let (Ω, F, (Ft)t≥0, P) be a complete filtered probability space. In this space, let f (.) and g(.) be bounded measurable functions mapping Rk into Rk and into the space of real k × n matrices, respectively. We now Consider the general n-dimensional stochastic system :

dXt = f (t, Xt)dt + g(t, Xt)dWt (2.3)

on t ≥ 0 with initial value X (0) = X0 , the solution is denoted by X (t, X0) : Assume that f (t, 0) = g(t, 0) = 0 for all t ≥ 0 Thus, we consider solutions to

$$X_t = X_0 + \int_0^t f(s, X_s)ds + \int_0^t g(s, X_s)dW_s \tag{2.4}$$

By a solution to the SIDE described by (2.4) we mean a continuous. Ft -adapted process X (.) which satisfies (2.4) with probability one.

If the following assumptions are satisfied

- The functions f, g, are measurable.

For every $t \in [0, T]$, $x, x' \in \mathbb{R}^n$, and there exists two constants K_1, K_2 such that,

- $\|f(t,x) - f(t,x')\| + \|g(t,x) - g(t,x')\| \leq K_1(\|x - x'\|)$
- $\|f(t,x)\|^2 + \|g(t,x)\|^2 \leq K_1^2(1 + \|x\|^2)$
- X_0 are square-integrable.

Then the solution of the system (2.3) is unique, for more details see [3].

Remark 2.1. If we Taking $X = (S, I, R)$,

$$f = \left(r_c S\left(1 - \frac{S}{k}\right) - \frac{\alpha SI}{1 + aI}, \frac{\alpha SI}{1 + aI} - \gamma I, \gamma I\right),$$

$$g = \left(\sigma_S F(S, I, R), \sigma_I G(S, I, R), \sigma_R H(S, I, R)\right),$$

then the SDE (2.2) has an strong unique solution.

2.3. *Proof*

Case 1:

Let's consider $t \in [0,T]$,
$X = (S,I,R)$, $X' = (S',I',R') \in \Omega =]s_1,s_2[\times]i_1,i_2[\times]r_1,r_2[$ with s_1,s_2,i_1,i_2,r_1 and r_2 are the constants, and we defined $\|X\| = \sqrt{\sum S^2+I^2+R^2}$
We take $F(S,I,R)=S, G(S,I,R)=I$ and $H(S,I,R)=R$.

We have $g=(\sigma_S S,\sigma_I I,\sigma_R R)$. So:

$$\| g(t,X)-g(t,X') \| \leq K \; \| X-X' \|$$

With $K = max\left\{(\sigma_S^2,\sigma_I^2,\sigma_R^2)\right\}$
Let's consider $f=(f_1,f_2,f_3)$ With:

$$f_1 = r_c S\left(1-\frac{S}{k}\right)-\frac{\alpha SI}{1+aI}$$
$$f_2 = \frac{\alpha SI}{1+aI}-\gamma I$$
$$f_3 = \gamma I$$

$$\|f_1(t,X)-f_1(t,X')\| \leq max\left((r_c+\frac{2r_c s_2}{k}+\alpha),\alpha\frac{s_2}{(1+ai_1)^2}\right)\|X-X'\|$$

$$\|f_2(t,X)-f_2(t,X')\| \leq max\left(\alpha,\alpha\frac{\alpha s_2}{(1+ai_1)^2}+\gamma\right)\|X-X'\|$$

So

$$\|f(t,X)-f(t,X')\|+\|g(t,X)-g(t,X')\| \leq K_1(\|X-X'\|)$$

With

$$K_1 = max\left\{\left((r_c+\frac{2r_c s_2}{k}+\alpha),\alpha\frac{s_2}{(1+ai_1)^2}\right),\left(\alpha,\alpha\frac{\alpha s_2}{(1+ai_1)^2}+\gamma\right),K\right\}$$

For the second inequality we have:

$$\|f_3(t,X)\|^2 \leq \gamma^2 i_2^2$$

$$\|f_1(t,X)\|^2 \leq \left(r_c s_2\left(1+\frac{s_2}{k}\right)+\frac{\alpha s_2 i_2}{1+ai_1}\right)^2$$
$$\|f_2(t,X)\|^2 \leq \left(\frac{\alpha s_2 i_2}{1+ai_1}+\gamma i_2\right)^2$$

Let's consider

$$K' = max\left\{\left(r_c s_2\left(1+\frac{s_2}{k}\right)+\frac{\alpha s_2 i_2}{1+ai_1}\right)^2, \left(\frac{\alpha s_2 i_2}{1+ai_1}+\gamma i_2\right)^2\right\}$$

And

$$\|g(t,X)\|^2 \leq max(\sigma_S^2, \sigma_I^2, \sigma_R^2)\|X\|^2$$

So we have

$$\|f(t,X)\|^2 + \|g(t,X)\|^2 \leq K' + max(\sigma_S^2, \sigma_I^2, \sigma_R^2)\|X\|^2$$
$$\leq K_1^2 + K_1^2\|X\|^2$$

Case 2:

We take $F(S,I,R) = \exp(S), G(S,I,R) = \exp(I)$, and $H(S,I,R) = \exp(R)$.
We have $g = \Big(\sigma_S \exp(S), \sigma_I \exp(I), \sigma_R \exp(R)\Big)$.
We Know that:

$$\|g(t,X) - g(t,X')\| = \sqrt{\left(\sigma_S^2\Big(\exp(S)-\exp(S')\Big)^2 + \sigma_S^2\Big(\exp(I)-\exp(I')\Big)^2 + \sigma_I^2\Big(\exp(R)-\exp(R')\Big)^2\right)}$$

using the following classical relation of exponential function $\exp(y) \geq 1+y \quad \forall y \in \mathbb{R}$. We get,

$$\|\exp(S)-\exp(S')\| \leq \|\exp(S)\|\Big\|1-\exp(S'-S)\Big\|$$
$$\leq \exp(s_2)\|S-S'\|$$

by the same reasoning we obtain,

$$\|\exp(I)-\exp(I')\| \leq \exp(i_2)\|I-I'\|$$
$$\|\exp(R)-\exp(R')\| \leq \exp(r_2)\|R-R'\|$$

which shows that;

$$\|f(t,X)-f(t,X')\| + \|g(t,X)-g(t,X')\| \leq K_1^{'}(\|X-X'\|)$$

with

$$K_1^{'} = max\Bigg\{\Bigg((r_c + \frac{2r_c s_2}{k} + \alpha), \alpha\frac{s_2}{(1+ai_1)^2}\Bigg), \Bigg(\alpha, \alpha\frac{\alpha s_2}{(1+ai_1)^2+\gamma}\Bigg), K_{11}\Bigg\}$$

such that $K_{11} = Kmax\Bigg(\exp(i_2), \exp(s_2), \exp(r_2)\Bigg)$

For the second inequality we have:

$$\begin{aligned}
\|g(t,X)\|^2 &= \sigma_S^2\exp(2S) + \sigma_I^2\exp(2I) + \sigma_R^2\exp(2R) \\
&\leq K^2\Bigg[\exp(2S)\Bigg(1 + \exp(-2(S-I)) + \exp(-2(S-R))\Bigg)\Bigg] \\
&\leq K^2\exp(2s_2)\Bigg[1 + \frac{1}{(S-I+1)^2} + \frac{1}{(S-R+1)^2}\Bigg] \\
&\leq K^2\exp(2s_2)\Bigg[\frac{(S-I+1)^2(S-R+1)^2}{(S-I+1)^2(S-R+1)^2} + \frac{(S-R+1)^2+(S-I+1)^2}{(S-I+1)^2(S-R+1)^2}\Bigg] \\
&\leq K^2\exp(2s_2)\Bigg[(S-I+1)^2(S-R+1)^2 + (S-R+1)^2 + (S-I+1)^2\Bigg] \\
&\leq K^2\exp(2s_2)\Bigg[(s_2-i_1+1)^2(s_2-r_1+1)^2 + s_2^2 + 2 + 4s_2 + \|X\|^2\Bigg] \\
&\leq K_1'^2\|X\|^2
\end{aligned}$$

And we know that

$$\begin{aligned}
\|f(t,X)\|^2 &\leq K' \\
&\leq K_1'^2
\end{aligned}$$

So we obtain:

$$\|f(t,X)\|^2 + \|g(t,X)\|^2 \leq K_1'^2(2 + \|X\|^2)$$

So

$$\|f(t,X)\|^2 + \|g(t,X)\|^2 \leq K_2'^2(1 + \|X\|^2)$$

With $K_2'^2 \geq 2K_1'^2$

2.4. *endemic equilibrium*

The deterministic SIR system admits a unique endemic equilibrium $E^* = (S^*, I^*, R^*)$, with

$$\begin{aligned}
S^* &= \frac{\gamma(1+aI^*)}{\alpha} \\
I^* &= \frac{ak\alpha r_c - 2r_c a\gamma - \alpha^2 k + \sqrt{\Delta}}{2r_c a^2\gamma} \\
R^* &= \frac{\gamma}{\delta}I^*
\end{aligned}$$

$$\Delta = [ak\alpha r_c - 2r_c a\gamma - \alpha^2 k]^2 + 4r_c a^2\gamma(k\alpha r_c - r_c\gamma)$$

For more details, refer to [14] and references therein.

2.5. Numerical Simulation and Dynamics Comparison

In this study, we will construct and implement numerical methods for solving “Stochastic Differential Equations”

(SDEs) using MATLAB. An SDE is a differential equation in which one or more of the terms, and hence the solution itself, is a stochastic process. Studying stochastic processes requires a departure from the familiar deterministic setting of ordinary and partial differential equations, into a world where the evolution of a quantity has an inherent random component and where the expected behavior of this quantity can be described in terms of probability distributions.

Depending on the mathematical model, we have a number of numerical techniques at our disposal. Using the Milstein’s higher order method in [2] to find the strong solution of system (2.2) with given initial value and the values of parameters. There can be many different approaches on deriving SDE model from the deterministic model, which may lead to different dynamical outcomes. In this sub- section, we compare the dynamics of stochastic differential equations with large diffusion terms.

In the first time taking the coefficients F (S, I , R) = S, G(S, I , R) = I and H (S, I , R) = R, the corresponding discretization of model is :

$$\begin{cases} S_{i+1} = S_i + \left(r_c S_i \left(1 - \frac{S_i}{k}\right) - \frac{\alpha S_i I_i}{1 + aI_i} \right)\Delta t + \sigma_S S_i \sqrt{\Delta t}\xi_i + \frac{\sigma_S^2}{2} S_i(\xi_i^2 - 1)\Delta t \\ I_{i+1} = I_i + \left(\frac{\alpha S_i I_i}{1 + aI_i} - \gamma I_i \right)\Delta t + \sigma_I I_i \sqrt{\Delta t}\eta_i + \frac{\sigma_I^2}{2} I_i(\eta_i^2 - 1)\Delta t \\ R_{i+1} = R_i(1 - \delta)\Delta t + \gamma I_i \Delta t + \sigma_R R_i \sqrt{\Delta t}\zeta_i + \frac{\sigma_R^2}{2} R_i(\zeta_i^2 - 1)\Delta t \end{cases} \tag{2.5}$$

where ξi , ηi and ζi , i = 1, ..., n, are the Gaussian random variables N(0, 1).

2.5.1. Numerical Results

In this part, we give the obtained numerical results the Milstein’s higher order method implemented using Matlab software. The parameters used are represented in following Table.

Table 1. Values of the parameters

Parameter	k	r_c	a	α	γ	δ
Value	100	0.5	2.3	1.49	0.25	0.43

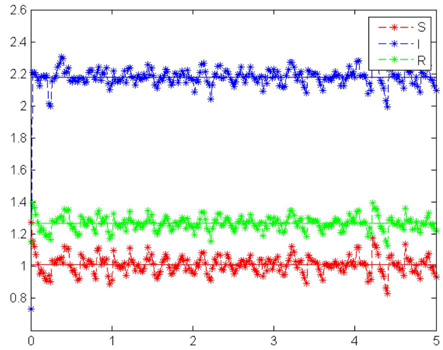

Figure 1: Solution of Stochastic Model

Figure 1 shows that the susceptible S, infective I and removed R population will be oscillating by the noise but around E∗ = (1.0093, 2.1807, 1.2679) with the noise intensities respectively σs = 0.12, σi = 0.1, σr = 0.21.

For σs = 2.72, σi = 1.89, σr = 1.51.

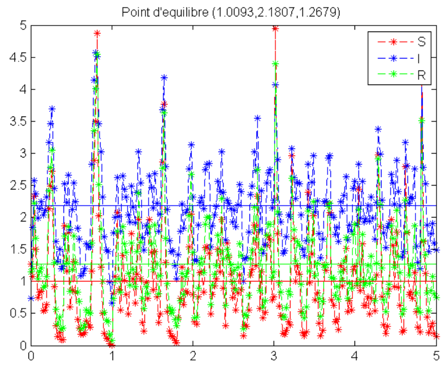

Figure 2: Solution of Stochastic Model

From Figure 2, one can see that with increasing the noise intensities, the solutions of model (2.2) will be oscillating strongly around the endemic point

E∗ = (1.0093, 2.1807, 1.2679) of model (2.6).

Remark 2.2. The effects of the intensity of noise levels From figures 1 and 2, we can conclude that, when the intensity of noise is small, the stochastic model preserves the property of the global stability. In this case, we can ignore noise and use the deterministic model to approximate the population dynamics. However, the large intensity of noise can force the solutions of model (2.2) to oscillate strongly around the disease-free. In these cases, we cannot ignore the effect of noise, therefore, we cannot use deterministic model but stochastic model to describe the population dynamics.

In this part we change the functions F, G and H in order to compare the fluctuation of System for the same values of σk , k = S, I , R. taking the coefficients F (S, I , R) = exp(S), G(S, I , R) = exp(I) and H (S, I , R) = exp(R), the corresponding discretization of model is :

$$\begin{cases} S_{i+1} = S_i + \left(r_c S_i\left(1 - \frac{S_i}{k}\right) - \frac{\alpha S_i I_i}{1 + aI_i}\right)\Delta t + \sigma_S \exp(S_i)\sqrt{\Delta t}\xi_i + \frac{\sigma_S^2}{2}\exp(S_i)(\xi_i^2 - 1)\Delta t \\ I_{i+1} = I_i + \left(\frac{\alpha S_i I_i}{1 + aI_i} - \gamma I_i\right)\Delta t + \sigma_I \exp(I_i)\sqrt{\Delta t}\eta_i + \frac{\sigma_I^2}{2}\exp(I_i)(\eta_i^2 - 1)\Delta t \\ R_{i+1} = R_i(1 - \delta)\Delta t + \gamma I_i \Delta t + \sigma_R \exp(R_i)\sqrt{\Delta t}\zeta_i + \frac{\sigma_R^2}{2}\exp(R_i)(\zeta_i^2 - 1)\Delta t \end{cases} \tag{2.6}$$

where ξi , ηi and ζi , $i = 1, ..., n$, are the Gaussian random variables $N(0, 1)$.

Taking the same values of $\sigma s = 0.12$, $\sigma i = 0.1$, $\sigma r = 0.21$ using in Figure 1, above in view to compare both figures.

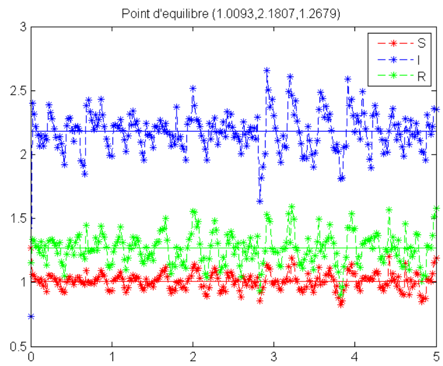

Figure 3: Solution of Stochastic Model

Figure 3 illustrate fluctuations around the equilibrium point $E* = (1.0093, 2.1807, 1.2679)$. In fact when varying functions F, G and H in order to enlarge the diffusion coefficients, then the fluctuation increases compare to figure1. When we take the same values $\sigma s = 2.72$, $\sigma i = 1.89$, $\sigma r = 1.51$ as figure 2, we have:

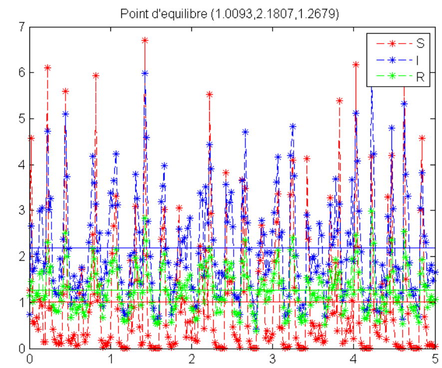

Figure 4: Solution of Stochastic

In this case we remark that the fluctuations around the equilibrium point has increased greatly in this figure.

3. CONCLUSIONS

In this paper, we extended the SIR epidemic model considered in [1]. In particular we analyze the epidemic model in a stochastic environment. The interest of this study lies in two aspects.

First, it presents the existence and uniqueness of solution using the Picard iteration method in the global behavior. Second, it performs some numerical simulations to illustrate the analytical results of stochastic model (2.2) by referring to

Milstein's higher order method, and then a comparative numerical study is done for different value of σ around the equilibrium epidemic. To study the effect of perturbation noise on the deterministic SIR model (2.1), we stochastically perturb model (2.6) with respect to white noise around its endemic equilibrium depending on the intensities of noise σ. We conclude when the intensities of noise are not sufficiently large, the population of the stochastic model may be stochastically permanent around of equilibrium epidemic.

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